

An old problem of Erdős: a graph without two cycles of the same length

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph on n vertices in which any two cycles are of different lengths. Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of simple graphs on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. In this paper, it is proved that for n sufficiently large, $mc(n) \leq \frac{15}{16}n$.

We make the following conjecture:

Conjecture.

$$\lim_{n \rightarrow \infty} \frac{mc(n)}{n} = 0.$$

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, P. Erdős raised the problem of determining $f(n)$ (see [1], p.247, Problem 11). Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of simple graphs on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. Let $sc(n)$ be the second-largest cycle length for all $G \in M_n$. Let $tc(n)$ be the third-largest cycle length for all $G \in M_n$. A natural question is what is the numbers of $mc(n)$,

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$sc(n)$, $tc(n)$. Let $mcn(n)$ be the maximum cycle numbers for all $G \in M_n$. A natural question is what is the numbers of $mcn(n)$. Let $b(n)$ be the maximum 2-connected block numbers for all $G \in M_n$. A natural question is what is the numbers of $b(n)$. Shi[23] proved that

Theorem 1 (Shi [23]).

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for $n \geq 3$ and $f(n) = f^*(n - 1) + 3$ for $n \geq 3$.

Lai[8] proved that

Theorem 2 (Lai [8]). For $n \geq e^{2m}(2m + 3)/4$,

$$f(n) < n - 2 + \sqrt{n \ln(4n/(2m + 3)) + 2n} + \log_2(n + 6).$$

Chen, Lehel, Jacobson and Shreve[3] gave a quick proof of this result.

Jia[6], Lai[7,8,9,10,11,12,13,14,15], Shi[23,24,25,26,27,28], Shi, Tang, Tang, Gong, Xu[29], Shi, Xu, Chen, Wang[30] obtained some additional related results.

Lai[16] proved that

Theorem 3 (Lai [16]).

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{40}{99}}.$$

and Lai[9] conjectured that

conjecture 4 (Lai [9]).

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

Boros, Caro, Füredi and Yuster[2] proved that

Theorem 5 (Boros, Caro, Füredi and Yuster[2]).

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

In 1988, Shi[23] proved that

Theorem 6 (Shi[23]). For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n - 15} - 3)]$.

In 1998, G. Chen, J. Lehel, M. S. Jacobson, and W. E. Shreve [3] proved that

Theorem 7 (Chen, Lehel, Jacobson and Shreve [3]). $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] improved this lower bound significantly:

Theorem 8 (Boros, Caro, Füredi and Yuster[2]). $f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.
and conjectured that

Conjecture 9 (Boros, Caro, Füredi and Yuster[2]). $\lim \frac{f_2(n)-n}{\sqrt{n}} = 1$.

It is easy to see that this Conjecture implies the (difficult) upper bound in the Erdős Turán Theorem [4][5](see [2]).

Markström [22] raised the problem

Problem 10 (Markström [22]). Determining the maximum number of edges in a Hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ be the maximum number edges in an n -vertex, Hamiltonian graph with no repeated cycle lengths. J. Lee, C. Timmons [18] proved the following.

Theorem 11 (J. Lee, C. Timmons [18]). If q is a power of a prime and $n = q^2 + q + 1$, then

$$g(n) \geq n + \sqrt{n - 3/4} - 3/2$$

A simple counting argument shows that $g(n) < n + \sqrt{2n} + 1$.

Let MH_n be the set of Hamiltonian graphs on n vertices and $g(n)$ edges in which any two cycles are of different lengths. Let $mcn_H(n)$ be the maximum cycle numbers for all $G \in MH_n$. A natural question is what is the numbers of $mcn_H(n)$.

J. Ma, T. Yang [21] proved that

Theorem 12 (Ma, Yang [21]). Any n -vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.

Let $f_2(n, k)$ be the maximum number of edges in a graph G on n vertices in which no two cycles have the same length and G which consists of k 2-connected blocks. A natural question is what is the maximum number of edges $f_2(n, k)$. It is clearly that $f_2(n, 1) = f_2(n)$.

By theorem 5, it is clearly that

$$f_2(n, k) \leq f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

H. Lin, M. Zhai, Y. Zhao [19] proved that

Theorem 13 (Lin, Zhai,Zhao [19]). Let G be a graph of order $n \geq 26$. If $\rho(G) \geq \rho(K_{1,n-1}^+)$, then G contains two cycles of the same length unless $G \cong K_{1,n-1}^+$.

and asked the following problem.

Problem 14 (Lin, Zhai,Zhao [19]). What is the maximum spectral radius among all 2-connected n -vertex graphs without two cycles of the same length?

Y. Shi [27]proved that

Theorem 15 (Shi [27]).

$$b(n) \leq [(\sqrt{8n+1} - 5)/2] + 1$$

C. Lai [7] proved that

Theorem 16 (Lai [7]). $mc(n) \leq n - 1$ for $n \geq \sum_{i=1}^{71} i - 8 \times 18$.

Survey papers on this problem can be found in Tian[31], Zhang[32], Lai and Liu[17].

The progress of all 50 problems in [1] can be found in Locke[20]. Let $v(G)$ denote the number of vertices, and $\varepsilon(G)$ denote the number of edges. In this paper, it is proved that

Theorem 17. For n sufficiently large,

$$mc(n) \leq \frac{15}{16}n.$$

2 Proof of the theorem 17

Proof. If $mc(n) > \frac{15}{16}n$, for n sufficiently large, then there is a simple graph G on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths, the maximum cycle length of G is $mc(n)$. Let G_1 be the block contain the cycle with length $mc(n)$. It is clear that $v(G_1) > \frac{15}{16}n$. By the result of Ma and Yang [21], $\varepsilon(G_1) \leq v(G_1) + \sqrt{v(G_1)} + o(\sqrt{v(G_1)})$. By the result of Boros, Caro, Füredi and Yuster [2], $\varepsilon(G) \leq v(G_1) + \sqrt{v(G_1)} + o(\sqrt{v(G_1)}) + V(G) - V(G_1) + 1 + 1.98\sqrt{V(G) - V(G_1) + 1}(1 + o(1)) \leq n + 1 + \sqrt{n} + o(\sqrt{n}) + 1.98\sqrt{\frac{1}{16}n}(1 + o(1)) \leq n + \frac{3}{2}\sqrt{n}$, for n sufficiently large. By the result of Shi [23] and Lai [16], $\varepsilon(G) = f^*(n) = f(n+1) - 3 > n + (\sqrt{2 + \frac{40}{99}} - o(1))\sqrt{n}$, for n sufficiently large. Note that $\varepsilon(G) \leq n + \frac{3}{2}\sqrt{n}$, this contradiction completes the proof.

It is clear that $mcn(n) \leq mc(n) - 2$.

By theorem 3, it is clearly that

$$mcn(n) \geq \sqrt{2 + \frac{40}{99}}\sqrt{n}(1 - o(1)).$$

We make the following conjecture:

Conjecture.

$$\lim_{n \rightarrow \infty} \frac{mc(n)}{n} = 0.$$

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References

- [1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976).
- [2] E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, *J. Combin. Theory Ser. B* 82(2001), 270-284. MR1842115
- [3] G. Chen, J. Lehel, M. S. Jacobson, and W. E. Shreve, Note on graphs without repeated cycle lengths, *J. Graph Theory* 29(1998), 11-15. MR1633908
- [4] P. Erdős, On a problem of Sidon in additive number theory and on some related problems. Addendum, *J. Lond. Math. Soc.* 19 (1944), 208. MR0014111
- [5] P. Erdős and P. Turán, On a problem of Sidon in additive number theory, and on some related problems, *J. Lond. Math. Soc.* 16 (1941), 212–215. MR0006197
- [6] X. Jia, Some extremal problems on cycle distributed graphs, *Congr. Numer.* 121(1996), 216-222. MR1431994
- [7] C. Lai, On the Erdős problem, *J. Zhangzhou Teachers College(Natural Science Edition)* 3(1)(1989) 55-59. MR1025502
- [8] C. Lai, Upper bound and lower bound of $f(n)$, *J. Zhangzhou Teachers College(Natural Science Edition)* 4(1)(1990) 29,30-34. MR1078010
- [9] C. Lai, On the size of graphs with all cycle having distinct length, *Discrete Math.* 122(1993) 363-364. MR1246693
- [10] C. Lai, The edges in a graph in which no two cycles have the same length, *J. Zhangzhou Teachers College (Natural Science)* 8(4)(1994), 30-34. MR1336586
- [11] C. Lai, The numbers of edge of one class graphs, *J. of Zhangzhou Teachers College (Natural Science)* 12(2)(1999), 7-9, 20. MR1723698
- [12] C. Lai, A lower bound for the number of edges in a graph containing no two cycles of the same length, *Electron. J. Combin.* 8(2001), Note 9, 1 - 6. MR1877662
- [13] C. Lai, The number of edges of some graphs in which no two cycles have the same length, *J. of Zhangzhou Teachers College (Natural Science)* 15(1) (2002), 10-14. MR1895073
- [14] C. Lai, Graphs without repeated cycle lengths, *Australas. J. Combin.* 27(2003), 101-105. MR1955391
- [15] C. Lai, On the size of graphs without repeated cycle lengths, *Discrete Appl. Math.* 232 (2017), 226-229. MR3711962
- [16] C. Lai, On the number of edges in some graphs, *Discrete Appl. Math.* 283 (2020), 751-755. MR4114937

- [17] C. Lai, M. Liu, Some open problems on cycles, *J. Combin. Math. Combin. Comput.* 91 (2014), 51-64. MR3287706
- [18] J. Lee, C. Timmons, A note on the number of edges in a Hamiltonian graph with no repeated cycle length, *Australas. J. Combin.* 69(2)(2017), 286-291. MR3703029
- [19] H. Lin, M. Zhai, Y. Zhao, Spectral Radius, Edge-Disjoint Cycles and Cycles of the Same Length, *Electron. J. Combin.* 29(2) (2022), P2.1
- [20] S. C. Locke, Unsolved problems: <http://math.fau.edu/locke/Unsolved.htm>
- [21] J. Ma, T. Yang, Non-repeated cycle lengths and Sidon sequences, *Israel J. Math.* 245(2) (2021), 639-674. MR4358259
- [22] K. Markström, A note on uniquely pancyclic graphs, *Australas. J. Combin.* 44 (2009), 105-110. MR2527003
- [23] Y. Shi, On maximum cycle-distributed graphs, *Discrete Math.* 71(1988) 57-71. MR0954686
- [24] Y. Shi, Some problems of cycle length distribution, *J. Nanjing Univ. (Natural Sciences)*, Special Issue On Graph Theory, 27(1991), 233-234.
- [25] Y. Shi, The number of edges in a maximum cycle distributed graph, *Discrete Math.* 104(1992), 205-209. MR1172850
- [26] Y. Shi, On simple MCD graphs containing a subgraph homomorphic to K_4 , *Discrete Math.* 126(1994), 325-338. MR1264498
- [27] Y. Shi, An upper bound on the size of simple MCD-graphs, *Pure Appl. Math.* 10(1994), Special Issue, 210-216. MR1369978
- [28] Y. Shi, A class of almost uniquely pancyclic graphs, *J. Systems Sci. Math. Sci.* 26(4) (2006), 433-439. MR2270870
- [29] Y. Shi, Y. Tang, H. Tang, L. Gong, L. Xu, Two classes of simple MCD graphs, *Discrete geometry, combinatorics and graph theory*, 177-188, Lecture Notes in Comput. Sci., 4381, Springer, Berlin, 2007. MR2364761
- [30] Y. Shi, L. Xu, X. Chen, M. Wang, Almost uniquely pancyclic graphs, *Adv. Math. (China)* 35(5) (2006), 563-569. MR2410416
- [31] F. Tian, The progress on some problems in graph theory, *Qufu Shifan Daxue Xuebao Ziran Kexue Ban.* 1986, no. 2, 30-36. MR0865617
- [32] K. Zhang, Progress of some problems in graph theory, *J. Math. Res. Exposition* 27(3) (2007), 563-576. MR2349503